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States on Generalised Logics

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Abstract

Let $\mathscr{L}$ be a generalised logic (orthocomplemented weakly modular σ-lattice) and $\mathscr{S}$ the set of all states (probability measures) on it, endowed with the most important metric physically. It is proved here that there exists a real Banach space X which contains $\mathscr{S}$ as a closed convex subset, whose norm induces the metric of $\mathscr{S}$ and such that the linear envelope of $\mathscr{S}$ is dense in X. Also, X is an ordered topological vector space with a normal positive cone having a nonempty interior. Finally, we prove that every convex automorphism of $\mathscr{S}$ can be extended uniquely to a continuous unit normed linear one-one operator on X onto itself.

1. *Introduction*

Extensions of several fundamental results in quantum mechanics to the more general theory, where the logic $\mathscr{L}$ is an orthocomplemented weakly modular σ-lattice, have been lacking for some time. Rigorous investigations connected with particles, symmetries and scattering theory, for example are difficult to handle in this generalised formalism and further mathematical developments are, therefore, necessary. These are problems related to the group of convex automorphisms of the set of states $\mathscr{S}$, consisting of all the probability measures on $\mathscr{L}$, and also to the structure of $\mathscr{S}$. The difficulties do not present themselves in the special case of quantum theory.

Quantum logic $\mathscr{L}_q$ is much more restricted than the generalised one above. It satisfies extra conditions such as separability, atomicity and completeness. By Piron's theorem (Piron, 1964; Varadarajan, 1968, p. 184), $\mathscr{L}_q$ is isomorphic to the lattice of all the projection operators of a separable Hilbert space $\mathscr{H}$ over the reals, the complex numbers or the quaternions. The case of complex Hilbert space is always assumed. By Gleason's theorem (Gleason, 1957), the set $\mathscr{S}_q$ of quantum states is isomorphic to the convex set of von Neumann operators of unit trace on $\mathscr{H}$. Thus $\mathscr{L}_q$ and $\mathscr{S}_q$ are intimately related in conventional quantum theory. A symmetry operation in the generalised theory $(\mathscr{L}, \mathscr{S})$ induces a convex automorphism on $\mathscr{S}$. In the quantum case this becomes a unitary or antiunitary operator on $\mathscr{H}$ and subsequently the problems mentioned above become manageable (Jauch, 1968; Varadarajan, 1968).

Here this situation is remedied for the generalised formalism in a manner having certain similarities with the quantum case and such that subsequently full use of analysis can be taken advantage of. Motivated by the requirement of endowing $\mathscr{S}$ with an especially important metric topology we prove that there exists a real Banach space X including $\mathscr{S}$ as a closed convex subset, whose norm induces the metric of $\mathscr{S}$ and such that the linear span of $\mathscr{S}$ is dense in X. This is the first main theorem. Its proof depends on some elementary lemmas on signed measures on $\mathscr{L}$. Their properties turn out to be somewhat different from those over σ-algebras because of the in general nondistributive nature of the logic. The final theorem deals with the convex automorphisms of $\mathscr{S}$. Such an automorphism is shown to be extendable uniquely to a continuous unit-normed one-one linear operator on X onto itself which is an isometry on $\mathscr{S}$. The proof of the continuity part depends on an important property of X, namely that it is an ordered topological vector space having a normal† positive cone with nonempty interior. This is the content of another theorem.

The Banach space X (of the generalised states) plays now a role similar to that of $\mathscr{H}$ in the quantum case. The similarities between the corresponding representation theorems of the group of convex automorphisms of $\mathscr{S}$ are also obvious. Applications can now be carried out. For abstract scattering theory see Kronfli (1969).

2. *Functions on Logics*

This section contains some properties of functions and measures on $\mathscr{L}$. Standard lattice theoretic notation will be employed. Two elements a and b of an orthocomplemented lattice L are said to be *compatible*, written $a \leftrightarrow b$, if there exists a Boolean subalgebra of L containing a and b. The lattice L is *weakly modular* if the relation $a < b$ implies $a \leftrightarrow b$. From now on $\mathscr{L}$ denotes an orthocomplemented weakly modular σ-complete lattice, called *generalised logic.*

Definition 2.1

A *countably additive* function on $\mathscr{L}$ is a real-valued function p on $\mathscr{L}$ such that for any disjoint sequence (a_n) in $\mathscr{L}$

$$p(\bigvee_n a_n) = \sum_n p(a_n)$$

The function p is (finitely) *additive* if the above equation holds for any finite disjoint subset of $\mathscr{L}$. The function p is a *signed measure* if

(i) $-\infty < p(a) < \infty \qquad (a \in \mathscr{L})$,
(ii) $p(\varnothing) = 0$,
(iii) p is countably additive.

† Definition of a normal cone is given in Section 4.

A *proper state* or *probability measure* on $\mathscr{L}$ is a nonnegative measure p with $p(I)=1$. The set of all proper states will be denoted by $\mathscr{S}$.

The following lemmas are required in later sections.

Lemma 2.2

(i) *If $a, b \in \mathscr{L}$ then $a \leftrightarrow b$ if and only if $a \leftrightarrow b^{\perp}$.* (ii) *If $x \in \mathscr{L}$ and (a_n) a sequence in $\mathscr{L}$ with $x \leftrightarrow a_n$ for all n, then*

$$x \vee \bigwedge_n a_n = \bigwedge_n x \vee a_n \qquad \text{and} \qquad x \wedge \bigvee_n a_n = \bigvee_n x \wedge a_n$$

Proof: Elementary.

Lemma 2.3

Let p be an additive function on $\mathscr{L}$, $x, y \in \mathscr{L}$ and $x < y$. Then

$$p(y) - p(x) = p(y \wedge x^{\perp})$$

If p is also nonnegative then it is monotone.

Proof: Let $c = y \wedge x^{\perp}$. Then $c \perp x$. Also $x \leftrightarrow y$ and hence by Lemma 2.2 using the fact $x < y$,

$$x \vee c = (x \vee y) \wedge (x \vee x^{\perp}) = y$$

The first result follows by the additivity of p. If, furthermore, $p \geqslant 0$ on $\mathscr{L}$ then $p(y) \geqslant p(x)$ and hence monotone. ∎

Definition 2.4

The *upper* and *lower variations* at $a \in \mathscr{L}$ of any real function p on $\mathscr{L}$ are defined by

$$p^{\pm}(a) = \sup\{\pm p(x) : x < a\}$$

respectively. Its (*total*) *variation* is the function

$$|p|(a) = p^{+}(a) + p^{-}(a) \qquad (a \in \mathscr{L})$$

Lemma 2.5

Let p be an additive function on $\mathscr{L}$. Then

(i) $p = p^{+} - p^{-}$,
(ii) *$p^{\pm}$ and $|p|$ are monotone.*

Proof: Let $x, a \in \mathscr{L}$ and $x < a$. Then by (Lemma 2.3)

$$p(x) = p(a) - p(a \wedge x^{\perp})$$

Hence

$$p^{+}(a) = p(a) + \sup\{-p(a \wedge x^{\perp}) : x < a\}$$

Part (i) is proved if we show that $L_a = \{x : x < a\}$ and $L_a \ = \{a \wedge x^{\perp} : x < a\}$ are equal. Since $a \wedge x^{\perp} < a$ then $L_a' \subset L_a$. Conversely, let $y \in L_a$, then $y < a$. This can be written as $y = a \wedge x^{\perp}$ where $x = a \wedge y^{\perp} < a$, implying $y \in L_a'$ or $L_a \subset L_a'$. Thus $L_a = L_a'$ proving (i).

Now let $a < b$. Then

$$\{\pm p(x) : x < a\} \subset \{\pm p(x) : x < b\}$$

or $p^{\pm}(a) \leqslant p^{\pm}(b)$. Thus $p^{\pm}$ and hence $|p| = p^{+} + p^{-}$ are all monotone, proving (ii). ■

Remarks

Part (i) of the above Lemma corresponds to the Jordan decomposition and of course also holds for signed measures. If p is a signed measure on $\mathscr{L}$ then in general $p^{\pm}$ and $|p|$ are not measures. In fact a concrete example could be constructed where $p^{\pm}$ are not even additive.† For a σ-algebra all three are in fact measures and the reason for this not being so on $\mathscr{L}$ is that in general $\mathscr{L}$ need not be distributive. This is unfortunate, for otherwise we will be able to show that span ($\mathscr{S}$) is precisely the set of all signed measures on $\mathscr{L}$—a great simplification if true. However, as mentioned above, this is not the case.

3. *The Banach Space of States*

Most problems of interest to generalised quantum theory require an appropriate topology on $\mathscr{S}$. The most suitable one physically is the metric topology introduced in the following.

Lemma 3.1

The real-valued function ρ on $\mathscr{S} \times \mathscr{S}$ defined by

$$\rho(p, q) = \sup\{|p(a) - q(a)| : a \in \mathscr{L}\} \qquad (p, q \in \mathscr{S})$$

is a metric on $\mathscr{S}$, called the 'natural metric'.

Proof: Elementary since $\mathscr{S}$ consists of bounded functions. ■

The following is the first main result of this work.

Theorem 3.2

The set X_1 of all signed measures on $\mathscr{L}$ with finite variations is a real Banach space with the norm

$$\|p\| = |p|(I) \qquad (p \in X_1)$$

Furthermore, $\mathscr{S}$ is a closed convex subset of $(X_1, \|\cdot\|)$ with the norm above inducing the natural metric ρ.

† A. M. Gleason, private communication.

Proof: It is obvious that X_1 is a real linear space. Also from the definition of variation in Definition 2.4

$$\|p+q\| \leqslant \|p\| + \|q\|$$
$$\|\alpha p\| = |\alpha| \cdot \|p\|$$

for all $p, q \in X_1$ and $\alpha \in R$. By Lemma 2.5 for any $p \in X_1$ and $a \in \mathscr{L}$,

$$|p(a)| = |p^+(a) - p^-(a)| \leqslant |p|(a) \leqslant |p|(I) = \|p\|$$

Thus $\|p\| = 0$ only if $p = 0$, and so $\|\cdot\|$ is a norm.

Now let (p_n) be a Cauchy sequence in $(X_1, \|\cdot\|)$. Then for any $a \in \mathscr{L}$

$$|p_n(a) - p_m(a)| \leqslant |(p_n - p_m)|(a)$$
$$\leqslant \|p_n - p_m\|$$

and so $(p_n(a))$ is a real Cauchy sequence which is, therefore, convergent to $p(a)$ say. Let (b_i) be a disjoint sequence in $\mathscr{L}$ with $b = \bigvee_i b_i$. Then

$$\sum_i p(b_i) = \lim_n \sum_i p_n(b_i) = \lim_n p_n(b) = p(b)$$

Thus (p_n) converges to the signed measure p. Similarly we can prove that $(|p_n|(a))$ is a real Cauchy sequence and hence convergent. Therefore, (p_n) converges to p in X_1. Thus X_1 is a Banach space.

Obviously $\mathscr{S}$ is a convex subset of X_1. Also for any $p \in \mathscr{S}$, $p \geqslant 0$ giving $|p| = p = p^+$ and $p^- = 0$. The natural metric can, therefore, be written as

$$\rho(p,q) = \sup\{|[|p|(a) - |q|(a)]| : a \in \mathscr{L}\} \qquad (p, q \in \mathscr{S})$$

Also from Lemma 2.5, the variation of a signed measure is monotone and hence

$$\|r\| = \sup\{|r|(a) : a \in \mathscr{L}\} = \rho(r, 0) \qquad (r \in X_1)$$

This gives

$$\|p - q\| = \rho(p - q, 0) = \rho(p, q)$$

Thus the norm $\|\cdot\|$ induces the natural metric ρ on $\mathscr{S}$. To complete the proof it remains to show that $\mathscr{S}$ is closed. Let (p_n) be any sequence in $\mathscr{S}$ converging to p in X_1. It is enough to show that $p \in \mathscr{S}$. Since $p_n \in \mathscr{S}$ for each n, then

$$p_n(I) = 1 = |p_n|(I) = \|p_n\|$$

Thus

$$|[1 - |p|(I)]| = |[|p_n|(I) - |p|(I)]|$$
$$\leqslant \|p_n - p\| \to 0 \qquad (n \to \infty)$$

or

$$|p|(I) = 1 = \|p\|$$

Furthermore, $p_n \in \mathscr{S}$ implying that $p_n^- = 0$ and so for any $a \in \mathscr{L}$, noting that p^- is monotone,

$$p^-(a) = |p_n^-(a) - p^-(a)| \leqslant \|p_n - p\| \to 0 \qquad (n \to \infty)$$

giving $p^- = 0$. Hence for any $a \in \mathscr{L}$

$$0 \leqslant p^+(a) = p(a) = |p|(a) \leqslant \|p\| = 1$$

or $p(I) = 1$ implying that $p \in \mathscr{S}$. The proof of the theorem is now complete. ■

Definition 3.3

From now on E will denote the linear span of $\mathscr{S}$ over the reals and X the closure of E in $(X_1, \|\cdot\|)$. Since X is a closed subspace of X_1 then it is itself a Banach space with norm $\|\cdot\|$. We call X the *Banach space of states* on $\mathscr{L}$. Any $p \in X$ is called a *generalised state* and any $p \in \mathscr{S}$ a *proper state.*

Corollary 3.4

There exists a real Banach space $(X, \|\cdot\|)$ *including* $\mathscr{S}$ *as a closed convex subset whose norm induces the natural metric and such that* span $(\mathscr{S})$ *is dense in* X.

Note that a generalised state p not in $\mathscr{S}$ need not have any physical significance. The reason for the name given to X is due to the similar role it plays to the Hilbert space of states in the quantum case.

4. *The Convex Automorphisms of the Set of States*

This section is devoted to the representation theory of the group of convex automorphisms of $\mathscr{S}$ in terms of certain linear operators on X. Their importance is due to the fact that they contain the symmetry operations as well as the dynamical group of the system.

Definition 4.1

A *convex automorphism* of $\mathscr{S}$ is a one-one mapping $A: p \to Ap$ of $\mathscr{S}$ onto itself such that for any sequence (c_n) of nonnegative numbers with $\sum_n c_n = 1$ and any sequence (p_n) in $\mathscr{S}$.

$$A \sum_n c_n p_n = \sum_n c_n A p_n$$

These maps form a group denoted by $\mathrm{Aut}(\mathscr{S})$.

The proof of the representation theorem of $\mathrm{Aut}(\mathscr{S})$ depends on an important property of the linear span E of $\mathscr{S}$, namely that E is an ordered topological vector space with a generating normal positive cone with non-empty interior. *Normality* of the positive cone C of E means that there exists a neighbourhood basis of 0 for the topology of E consisting of sets F such that $F = (F + C) \cap (F - C)$. Implications of this ordering structure required in the sequel are contained in the following.

Theorem 4.2

With the partial ordering $p \leqslant q$ for p, $q \in X$ whenever $p(a) \leqslant q(a)$ for all $a \in \mathscr{L}$, X is an ordered Banach space with a positive cone $C = \{p \in X : p \geqslant 0\}$. Furthermore, C is closed, normal and has nonempty interior.

Proof: Clearly X is ordered by C. For any p, $q \in X$ and $0 \leqslant p \leqslant q$ using Lemma 2.5,

$$|p|(a) = p^+(a) = p(a) \leqslant q(a) = q^+(a) = |q|(a) \qquad (a \in \mathscr{L})$$

implying that $\|p\| \leqslant \|q\|$. This in turn implies that C is normal; see Peressini (1967, proposition (1.5) p. 63).

Note also that

$$C = \left\{\sum_i \alpha_i p_i : (\alpha_i) \text{ finite set of nonnegative numbers, } (p_i) \subset \mathscr{S}\right\}$$

and hence it is easy to prove that C is closed. Finally C is a closed subset of the complete normed space X and hence C is a complete metric space. By Baire's category theorem C is of second category. If the interior C^0 of C was empty then $\bar{C}^0 = C^0 = \varnothing$ implying that C is of first category which is a contradiction. ■

The following is the required representation theorem for $\mathrm{Aut}(\mathscr{S})$.

Theorem 4.3

Every $A \in \mathrm{Aut}(\mathscr{S})$ can be extended uniquely to a continuous unit normed linear one-one operator $\tilde{A}$ on X onto itself which is an isometry on C and such that

(i) $\tilde{A}^{-1} = \widetilde{A^{-1}}$.

Proof: We extend A by linearity to the linear span E of $\mathscr{S}$ and denote this extension by the same symbol A. Now, since $A(\mathscr{S}) = \mathscr{S}$ then it is clear that $AC \subset C$ and, therefore, A is a positive linear operator on $E \to E$. From Theorem 4.2, C is normal and $C^0 \neq \varnothing$ and A is positive, therefore, A is continuous on $E \to E$; see Peressini (1967, proposition (2.16) p. 86). Since $E = C - C$, C is not dense in E and hence a nonzero positive continuous map, such as A, always exists.

Thus every $A \in \mathrm{Aut}(\mathscr{S})$ is a continuous one-one linear map on E onto itself. Now A is a bounded linear transformation on the normed linear space $(E, \|\cdot\|)$ into the Banach space $(X, \|\cdot\|)$ and hence permits a closure $\tilde{A}$ extending A to the whole of the closure $\bar{E} = X$ of E. Furthermore, $\tilde{A}$ is unique, continuous and such that $\|\tilde{A}\| = \|A\|$ (Bachman & Narici, 1966, section 17.2). Result (i) follows trivially since $A^{-1} \in \mathrm{Aut}(\mathscr{S})$. Because of (i) $\tilde{A}$ is one-one onto with domain and range equal to X.

That A is an isometry on C follows from the fact that $p \in \mathscr{S}$ implies $\|p\| = |p|(I) = 1$ and also that any $r \in C$ is of the form $\sum_i \alpha_i p_i$ where (α_i) is a finite subset of $[0, \infty)$ and $(p_i) \subset \mathscr{S}$. Thus

$$\sum \alpha_i = \sum \alpha_i \|p_i\| \geqslant \|r\| = |r|(I) \geqslant |r(I)| = \left|\sum_i \alpha_i p_i(I)\right| = \sum \alpha_i$$

or

$$\|r\| = \sum \alpha_i$$

Furthermore, $Ap_i \in \mathscr{S}$ and hence again $\|Ar\| = \sum \alpha_i$.

Therefore, $\|Ar\| = \|r\|$ $(r \in C)$, implying that A and, therefore, $\tilde{A}$ are isometries on C. Finally, $\|A\| = 1$ follows trivially from the above. ∎

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References

Bachman, G. and Narici, L. (1966). *Functional Analysis*. Academic Press, New York.
Gleason, A. M. (1957). *Journal of Rational Mechanics and Analysis*, **6**, 885.
Jauch, J. M. (1968). *Foundations of Quantum Mechanics*. Addison-Wesley Publishing Company, Reading, Mass.
Kronfli, N. S. (1969). *International Journal of Theoretical Physics*, Vol. 2, No. 4, 345.
Peressini, A. L. (1967). *Ordered Topological Vector Spaces*. Harper and Row (Publishers), New York.
Piron, C. (1964). *Helvetica Physica Acta*, **37**, 439.
Varadarajan, V. S. (1968). *Geometry of Quantum Theory*, Vol. I. Van Nostrand Co. Inc., Princeton, N.J.